Lagrange Interpolation

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Let Q(x) be the Quadratic Formula generated from points a, b, c; where $d_1 < d_2 < d_3$ and $f(d_1) > f(d_2) < f(d_3)$, using Lagrange Interpolation:

$$Q(x) = f(d_1) \times \frac{(x-d_2)(x-d_3)}{(d_1-d_2)(d_1-d_3)} + f(d_2) \times \frac{(x-d_3)(x-d_1)}{(d_2-d_3)(d_2-d_1)} + f(d_3) \times \frac{(x-d_1)(x-d_2)}{(d_3-d_1)(d_3-d_2)}$$

$$\implies Q'(x) = f(d_1) \times \frac{(x-d_2)+(x-d_3)}{(d_1-d_2)(d_1-d_3)} + f(d_2) \times \frac{(x-d_3)+(x-d_1)}{(d_2-d_3)(d_2-d_1)} + f(d_3) \times \frac{(x-d_1)+(x-d_2)}{(d_3-d_1)(d_3-d_2)}$$

For minima, Q'(x) = 0

$$\therefore f(d_1) \times \frac{(x-d_2)+(x-d_3)}{(d_1-d_2)(d_1-d_3)} + f(d_2) \times \frac{(x-d_3)+(x-d_1)}{(d_2-d_3)(d_2-d_1)} + f(d_2) \times \frac{(x-d_1)+(x-d_2)}{(d_3-d_1)(d_3-d_2)} = 0$$

Multiplying by $(d_1-d_2)\cdot(d_2-d_3)\cdot(d_3-d_1)$:

$$f(d_1) \times (2x-d_2-d_3)(d_3-d_2) + f(d_2) \times (2x-d_3-d_1)(d_1-d_3) + f(d_3) \times (2x-d_1-d_2)(d_2-d_1) = 0$$

$$\implies 2x\{f(d_1)(d_3-d_2)+f(d_2)(d_1-d_3)+f(d_3)(d_2-d_1)\} - \{f(d_1)(d_3^2-d_2^2)+f(d_2)(d_1^2-d_3^2)+f(d_3)(d_2^2-d_1^2)\} = 0$$

Since we have considered Q'(x) = 0, we can now consider the x here as x_{min}

$$\therefore x_{min} = \frac{1}{2} \times \frac{f(d_1)(d_3^2-d_2^2)+f(d_2)(d_1^2-d_3^2)+f(d_3)(d_2^2-d_1^2)}{f(d_1)(d_3-d_2)+f(d_2)(d_1-d_3)+f(d_3)(d_2-d_1)}$$

$$\implies x_{min} = \frac{1}{2} \times \frac{f(d_1)(d_3-d_2)(d_3+d_2)+f(d_2)(d_1-d_3)(d_1+d_3)+f(d_3)(d_2-d_1)(d_2+d_1)}{f(d_1)(d_3-d_2)+f(d_2)(d_1-d_3)+f(d_3)(d_2-d_1)}$$

$$\implies x_{min} = d_2 + \frac{1}{2} \times \frac{f(d_1)(d_3-d_2)(d_3+d_2-2d_2)+f(d_2)(d_1-d_3)(d_1+d_3-2d_2)+f(d_3)(d_2-d_1)(d_2+d_1-2d_2)}{f(d_1)(d_3-d_2)+f(d_2)(d_1-d_3)+f(d_3)(d_2-d_1)}$$

$$\implies x_{min} = d_2 + \frac{1}{2} \times \frac{f(d_1)(d_3-d_2)(d_3+d_2-2d_2)+f(d_2)(d_1-d_3)(d_1+d_3-2d_2)+f(d_3)(d_2-d_1)(d_2+d_1-2d_2)}{f(d_1)(d_3-d_2)+f(d_2)(d_1-d_3)+f(d_3)(d_2-d_1)}$$

$$\implies x_{min} = d_2 + \frac{1}{2} \times \frac{f(d_1)(d_3-d_2)^2+f(d_2)(d_1^2+d_1d_3-2d_1d_2-d_1d_3-d_3^2+2d_2d_3+d_2^2-d_2^2)-f(d_3)(d_2-d_1)^2}{f(d_1)(d_3-d_2)+f(d_2)(d_1-d_3+d_2-d_2)+f(d_3)(d_2-d_1)}$$

$$\implies x_{min} = d_2 + \frac{1}{2} \times \frac{f(d_1)(d_3-d_2)^2+f(d_2)\{(d_2-d_1)^2-(d_3-d_2)^2\}-f(d_3)(d_2-d_1)^2}{f(d_1)(d_3-d_2)+f(d_2)(d_1-d_3+d_2-d_2)+f(d_3)(d_2-d_1)}$$

$$\implies x_{min} = d_2 + \frac{1}{2} \times \frac{\{f(d_1)-f(d_2)\}(d_3-d_2)^2-\{f(d_3)-f(d_2)\}(d_2-d_1)^2}{\{f(d_1)-f(d_2)\}(d_3-d_2)+\{f(d_3)-f(d_2)\}(d_2-d_1)}$$

Let point d_1, d_2, d_3 be equidistant; meaning $(d_2-d_1) = (d_3-d_2) = z$:

$$\implies x_{min} = d_2 + \frac{1}{2} \times \frac{\{f(d_1)-f(d_3)\}}{f(d_1)+f(d_3)-2f(d_2)} \times z$$

Let $f(d_1) = C_1$, $f(d_2) = C_2$, $f(d_3) = C_3$:

$$\implies x_{min} = d_2 + \frac{1}{2} \times \frac{C_1-C_3}{C_1+C_3-2C_2} \times z$$

If x_{min} is considered d_{est} and z = 1, we have:

$$d_{est} = d_2 - \frac{1}{2} \times \frac{C_3-C_1}{C_1+C_3-2C_2}$$